

# PRICING INSURANCE DRAWDOWN-TYPE CONTRACTS WITH UNDERLYING LÉVY ASSETS

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**ABSTRACT.** In this paper we consider some insurance policies related with drawdown and drawup events of log-returns for an underlying asset modeled by a spectrally negative geometric Lévy process. We consider four contracts among which three were introduced in [16] for a geometric Brownian motion. The first one is an insurance contract where protection buyer pays a constant premium until the drawdown of fixed size of log-returns occurs. In return he/she receives certain insured amount at the drawdown epoch. Next insurance contract provides protection from any specified drawdown with a drawup contingency. This contract expires early if certain fixed drawup event occurs prior to fixed drawdown. The last two contracts are extensions of the previous ones by additional cancellable feature which allows an investor to terminate the contract earlier. We focus on two problems: calculating the fair premium  $p$  for the basic contracts and identifying the optimal stopping rule for the policies with cancellable feature. To do this we solve some two-sided exit problems related with the drawdown and the drawup of spectrally negative Lévy processes which is of own scientific interest. We also heavily rely on a theory of optimal stopping.

**KEYWORDS.** insurance contract  $\star$  fair valuation  $\star$  drawdown  $\star$  drawup  $\star$  Lévy process  $\star$  optimal stopping.

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## 1. INTRODUCTION

The drawdown of a given process is the distance of the current value away from the maximum value it has attained to date. Similarly, the drawup of a given process is defined as the current rise of the present value over the running minimum. Both of them have been customarily used as dynamic risk measures. In fact, drawdown process does not only provide dynamic measures of risk, but can also be viewed as measures of relative regret. Similarly drawup process can be viewed as measures of relative satisfaction. Thus, a drawdown or a drawup of a certain number may signal the time when investor may choose to change his/her investment position, which depends on his/her perception of future moves of the market and his/her risk aversion.

The interest in drawdown asset has been strongly raised by recent financial crisis. Large market drawdown may bring portfolio losses, liquidity shocks and even future recessions. Therefore risk management of drawdown has become so important among practitioners; see e.g. [2] for portfolio optimization under constraints on the drawdown process, [1, 5] for the distribution of the maximum drawdown of drifted Brownian motion and time-adjusted measure of performance known as the Calmar ratio and [11, 13, 14] for drawdown process as dynamic measure of risk. For an overview of the existing techniques for analysis of market crashes as well as a collection of empirical studies of the drawdown process and the maximum drawdown process, please refer to Sornette [12].

It is then natural that fund managers have strong incentive to seek insurance against drawdown. In fact, as works of [1, 13, 14] argue, some market-traded contracts, such as vanilla and look-back puts, have only limited ability to insure the market drawdown. Therefore the drawdown protection can be useful also for individual investors.

In this paper we follow Zhang et al. [16] pricing some insurance contracts against drawdown (and drawup) events of log-returns of stock price modeled by exponential Lévy process and identifying the optimal stopping rules. We also identify for these contracts so-called fair premium rates for which contracts prices equal zero.

In its simplest form, the first drawdown insurance contract involves a continuous premium payment by the investor (protection buyer) to insure a drawdown of log-returns of the underlying asset over a pre-specified level. Possible buyer of this insurance contract might think that it is unlikely to get large drawdown and he/she might want to stop paying the premium. Therefore we expand the simplest contract by adding cancellable feature. In this case, the investor receives right to terminate the contract earlier and in this case he/she pay penalty for doing so. We show that the investor's optimal cancellable time is based on the first passage time of the drawdown of log-return process.

Moreover, we also consider a related insurance contract that protects the investor from a drawdown of log-return of the asset price preceding a drawup related with it. In other words, the insurance contract expires early if a drawup event occurs prior to a drawdown. From the investors perspective, when a drawup is realized, there is little need to insure against a drawdown. Therefore, this drawup contingency automatically stops the premium payment and it is an attractive feature that could potentially reduce the cost of the drawdown insurance. Finally, we also added cancellable feature to this contract.

Zhang et al. [16] considered only a risky asset modeled by the geometric Brownian motion. However, in recent years, empirical study of financial data reveals the fact that the distribution of the log-return of stock price exhibits features which cannot be captured by the normal distribution such as heavy tails and asymmetry. For the purpose of replicating more effectively these features and for reproducing a wide variety of implied volatility skews and smiles, there has been a general shift in the literature to modeling a risky asset with an exponential Lévy process as an alternative to exponential of a linear Brownian motion; see Kyprianou [3] and Øksendal and Sulem [8] for overviews. Therefore looking for a better fitting of the evolution of the stock price process to the real data, in this paper we price derivative securities in market by a general geometric spectrally negative Lévy process. That is, logarithm of a risky asset in our case will be a process with stationary and independent increments with no positive jumps.

The last contract analyzed in this paper taking into account drawdown and drawup with cancellable feature is considered for the first time in literature. At the same, although it is the most complex one, it produces very interesting and surprising results. In particular, we discover new phenomenon for optimal stopping contract rule in this contract. In the phenomenon, the investor's stopping rule is also at a first passage time of the drawdown of log-returns process, similarly like it is for the second

contract without drawup contingency. Still, the level of termination is different taking into account drawup event.

Our approach is based on the classical fluctuation theory for the spectrally negative Lévy processes (related with so-called scale functions) and some new exit identities for reflected Lévy processes. The latter ones identify two-sided exit problems for drawup and drawdown first passage times. A key element of our approach is path analysis and using some results of Mijatović and Pistorius [6]. We also heavily use optimal stopping theory. In a market where the underlying dynamics for the stock price process is driven by the exponential of a linear Brownian motion the valuation is transformed into a free boundary problem. However, by allowing jumps to appear in the sample paths of the underlying dynamics of the stock price process, this idea breaks down. To tackle these infinite horizon problem we use the so-called „guess and verify” method. For this method, one guesses what the optimal value function and optimal stopping should be and then tries to verify that this candidate solution is indeed the optimal one by putting it through a verification theorem. What is meant by the latter is that the value function identified by guessed stopping rule applied to the log-return price process constructs a smallest, in some sense, discounted supermartingale.

In this paper we also analyze many particular examples and give extensive numerical analysis showing the dependence of the contract and stopping time on the model’s parameters. We focus mainly on the case when logarithm of the asset price is a linear Brownian motion or drift minus compound Poisson process (so-called Cramér-Lundberg risk process).

The paper is organized as follows. In Section 2 we introduce main definitions, notations and main fluctuation identities. We analyze insurance contracts based on drawdown and additional drawup in Sections 3 and 4, respectively. We finish our paper by the numerical analysis performed in Section 5 and Conclusions in Section 6.

## 2. PRELIMINARIES

We work on complete filtered probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  satisfying the usual conditions. We model a logarithm of the underlying risky asset price  $\log S_t$  by a spectrally negative Lévy process  $X_t$ , that is  $S_t = \exp\{X_t\}$  is a geometric Lévy process. This means that  $X_t$  is a stationary stochastic process with independent increments, right-continuous paths with left limits and has only negative jumps.

Many identities will be given in terms of so-called scale functions which are defined in following way. We start from defining so-called Laplace exponent of  $X_t$ :

$$(1) \quad \psi(\phi) = \log \mathbb{E}[e^{\phi X_1}]$$

which is well defined for  $\phi \geq 0$  due to the absence of positive jumps. Recall that by Lévy-Khintchine theorem:

$$(2) \quad \psi(\phi) = \mu\phi + \frac{1}{2}\sigma^2\phi^2 + \int_{(0,\infty)} (e^{-\phi u} - 1 + \phi u \mathbb{1}_{(u < 1)}) \Pi(du),$$

which is analytic for  $\Im m(\phi) \leq 0$ , where  $\mu$  and  $\sigma \geq 0$  are real and  $\Pi$  is so-called Lévy measure. It is easy to observe that  $\psi$  is zero at the origin, tends to infinity at infinity and is strictly convex. We denote by  $\Phi : [0, \infty) \rightarrow [0, \infty)$  the right continuous inverse of  $\psi$  so that it satisfies the following:

$$\Phi(r) = \sup\{\phi > 0 : \psi(\phi) = r\} \quad \text{and} \quad \psi(\Phi(r)) = r \quad \text{for all } r \geq 0.$$

For  $r \geq 0$  we define a continuous and strictly increasing function  $W^{(r)}$  on  $[0, \infty)$  with the Laplace transform given by:

$$(3) \quad \int_0^\infty e^{-\phi u} W^{(r)}(u) du = \frac{1}{\psi(\phi) - r},$$

where  $\psi$  is a Laplace exponent of  $X_t$  given in (1). It is so-called the first scale function. The second one is related with the first one via the following relationship:

$$(4) \quad Z^{(r)}(u) = 1 + r \int_0^u W^{(r)}(\phi) d\phi.$$

In this paper we will assume that

$$(5) \quad W^{(r)} \in \mathcal{C}^1(\mathbb{R}_+)$$

for  $\mathbb{R}_+ = [0, \infty)$ . This assumption is satisfied when the process  $X_t$  has non-trivial gaussian component or it is of unbounded variation or the jumps have the density; see [4, Lem. 2.4]. The scale functions are used in two-sided exit formulas:

$$(6) \quad \mathbb{E}_x \left[ e^{-r\tau_a^+}; \tau_a^+ < \tau_0^- \right] = \frac{W^{(r)}(x)}{W^{(r)}(a)},$$

$$(7) \quad \mathbb{E}_x \left[ e^{-r\tau_0^-}; \tau_0^- < \tau_a^+ \right] = Z^{(r)}(x) - Z^{(r)}(a) \frac{W^{(r)}(x)}{W^{(r)}(a)},$$

where  $x \leq a$ ,  $r \geq 0$  and

$$(8) \quad \tau_a^+ = \inf\{t \geq 0 : X_t \geq a\}, \quad \tau_a^- = \inf\{t \geq 0 : X_t \leq a\}$$

are the first passage times. We also used the following notational convention:  $\mathbb{E}[\cdot \mathbb{1}_{\{A\}}] = \mathbb{E}[\cdot; A]$  for any event  $A$ .

Let us denote:

$$\overline{X}_t = \sup_{s \leq t} X_s, \quad \underline{X}_t = \inf_{s \leq t} X_s.$$

In this paper, we analyze some insurance contracts related with the drawdown and drawup processes of log-return of the asset price  $S_t$ , that is, with the drawdown and drawup processes of  $X_t$ . The drawdown is the difference between running maximum of the underlying process and its current value and the drawup is difference between process current value and its running minimum. Here, we additionally assume that the drawdown and drawup processes start from some points  $y > 0$  and  $z > 0$ , respectively. That is,

$$(9) \quad D_t = \overline{X}_t \vee y - X_t, \quad U_t = X_t - \underline{X}_t \wedge (-z).$$

Above the values  $y$  and  $-z$  can be interpreted as historical maximum and historical minimum of process  $X$ . The crucial for further work are the following first passage times of the drawdown process and the drawup process, respectively:

$$(10) \quad \tau_D^+(a) = \inf\{t \geq 0 : D_t \geq a\}, \quad \tau_D^-(a) = \inf\{t \geq 0 : D_t \leq a\},$$

$$(11) \quad \tau_U^+(a) = \inf\{t \geq 0 : U_t \geq a\}, \quad \tau_U^-(a) = \inf\{t \geq 0 : U_t \leq a\}.$$

Later, we will use the following notational convention:

$$\mathbb{P}_{|y}[\cdot] := \mathbb{P}[\cdot | D_0 = y], \quad \mathbb{P}_{|y|z}[\cdot] := \mathbb{P}[\cdot | D_0 = y, U_0 = z], \quad \mathbb{P}_{x|y|z}[\cdot] := \mathbb{P}[\cdot | X_0 = x, D_0 = y, U_0 = z].$$

Finally, we denote  $\mathbb{P}_x[\cdot] := \mathbb{P}[\cdot | X_0 = x]$  with  $\mathbb{P} = \mathbb{P}_0$  and  $\mathbb{E}_{|y}, \mathbb{E}_{|y|z}, \mathbb{E}_{x|y|z}, \mathbb{E}_x, \mathbb{E}$  are the corresponding expectations to above measures. We finish this section with two main formulas (the first one is given in Mijatović and Pistorius [6, Thm.4] and the second one follows from (6)) that identify the joint laws of  $\{\tau_U^+, \overline{X}_{\tau_U^+}, \underline{X}_{\tau_U^+}\}$  and  $\{\tau_D^-(\theta), \underline{X}_{\tau_D^-(\theta)}\}$ :

$$(12) \quad \mathbb{E} \left[ e^{-r\tau_U^+(b) + u\underline{X}_{\tau_U^+(b)}}; \overline{X}_{\tau_U^+(b)} < v \right] = e^{ub} \frac{1 + (r - \psi(u)) \int_0^{b-v} e^{-uy} W^{(r)}(y) dy}{1 + (r - \psi(u)) \int_0^b e^{-uy} W^{(r)}(y) dy} - e^{-u(b-v)} \frac{W^{(r)}(b-v)}{W^{(r)}(b)},$$

$$(13) \quad \mathbb{E}_{|y} \left[ e^{-r\tau_D^-(\theta)}; \underline{X}_{\tau_D^-(\theta)} > -x \right] = \mathbb{E}_x \left[ e^{-r\tau_{y-\theta+x}^+}; \tau_{y-\theta+x}^+ < \tau_0^- \right] = \frac{W^{(r)}(x)}{W^{(r)}(y - \theta + x)}.$$

### 3. DRAWDOWN INSURANCE CONTRACT

**3.1. Fair premium.** In this section, we consider the insurance contract in which protection buyer pays a constant premium  $p \geq 0$  continuously until the drawdown of log-returns of the asset price of size  $a > 0$  occurs. In return she/he receives the insured amount  $\alpha \geq 0$  at the drawdown epoch. Let  $r \geq 0$  be the risk-free interest rate. The contract price is equal to the discounted value of the future cash-flows:

$$(14) \quad f(y, p) = \mathbb{E}_{|y} \left[ - \int_0^{\tau_D^+(a)} e^{-rt} p dt + \alpha e^{-r\tau_D^+(a)} \right].$$

Note that in this contract the investor wants to protect herself/himself from the asset price  $S_t = e^{X_t}$  falling down from the previous maximum more than fixed level  $e^a$  for some  $a > 0$ . In other words, she/he believes that even if the price will go up again after the first drawdown of size  $e^a$  it will not bring her/him sufficient profit. Therefore, she/he is ready to take this type of contract to reduce loss by getting  $\alpha > 0$  at the drawdown epoch.

Note that

$$(15) \quad f(y, p) = \left( \frac{p}{r} + \alpha \right) \xi(y) - \frac{p}{r},$$

where

$$(16) \quad \xi(y) := \mathbb{E}_{|y} \left[ e^{-r\tau_D^+(a)} \right]$$

is the conditional Laplace transform of  $\tau_D^+(a)$  given that  $D_0 = y \in (0, a)$ . To price the contract (14) we start from identifying the crucial function  $\xi$ .

**Proposition 1.** *The conditional Laplace transform  $\xi(\cdot)$  is given by*

$$(17) \quad \xi(y) = Z^{(r)}(a - y) - rW^{(r)}(a - y) \frac{W^{(r)}(a)}{W^{(r)}(a)}.$$

*Proof.* Note that  $\tau_D^-(0)$  is the first time that the drawdown process  $D_t$  pass the level 0, which means that the process  $X_t$  attains its historical maximum. It is done in a continuous way by assumed spectral negativity of Lévy process  $X$ . By the strong Markov property of process  $D_t$  at  $\tau_D^-(0)$  we have that

$$\begin{aligned} \xi(y) &= \mathbb{E}_{|y} \left[ e^{-r\tau_D^+(a)} \right] \\ &= \mathbb{E}_{|y} \left[ e^{-r\tau_D^+(a)} ; \tau_D^+(a) < \tau_D^-(0) \right] + \mathbb{E}_{|y} \left[ e^{-r\tau_D^-(0)} ; \tau_D^-(0) < \tau_D^+(a) \right] \xi(0) \\ &= \mathbb{E}_{a-y} \left[ e^{-r\tau_0^-} ; \tau_0^- < \tau_a^+ \right] + \mathbb{E}_{a-y} \left[ e^{-r\tau_a^+} ; \tau_a^+ < \tau_0^- \right] \xi(0) \\ (18) \quad &= Z^{(r)}(a - y) - Z^{(r)}(a) \frac{W^{(r)}(a - y)}{W^{(r)}(a)} + \frac{W^{(r)}(a - y)}{W^{(r)}(a)} \xi(0), \end{aligned}$$

where the third equation follows from the two-sided exit formulas given in (6) - (7). Therefore the problem of finding  $\xi$  is reduced to identifying  $\xi(0)$ . The latter one can be obtained from [10, Prop. 2(ii), p. 191]:

$$(19) \quad \xi(0) = Z^{(r)}(a) - rW^{(r)}(a) \frac{W^{(r)}(a)}{W^{(r)}(a)}.$$

This completes the proof. □

Thus we have the following theorem.

**Theorem 1.** *The value of the contract (14) is given in (15) for  $\xi$  identified in (17).*

The fair situation for both sides, the insurance company and investor, is when contract price at conclusion moment equals 0. We say then that the premium  $p^*$  is fair when

$$f(y, p^*) = 0.$$

From (15) using Proposition 1 we derive the following theorem.

**Theorem 2.** *For the contract (14) the fair premium equals:*

$$(20) \quad p^* = \frac{r\alpha\xi(y)}{1 - \xi(y)}.$$

**3.2. Cancellable feature.** We now extend the previous contract by cancellable feature. In other words, we give the investor right to terminate the contract by paying fixed fee  $c \geq 0$  at any time prior to a pre-specifies drawdown of log-return of the asset price of size  $a > 0$ . This contract is addressed to the investors who are not willing to pay premium any longer after they stopped to believe that a large drawdown of the asset price may happen. The contract value equals then:

$$(21) \quad F(y, p) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_{|y} \left[ - \int_0^{\tau_D^+(a) \wedge \tau} e^{-rt} p dt - ce^{-r\tau} \mathbb{1}_{(\tau < \tau_D^+(a))} + \alpha e^{-r\tau_D^+(a)} \mathbb{1}_{(\tau_D^+(a) \leq \tau)} \right],$$

where  $\mathcal{T}$  is a family of all  $\mathcal{F}_t$ -stopping times.

One of the main goals of this paper is to identify the optimal stopping rule  $\tau^*$  that realizes the price  $F(y, p)$ . We start from the simple observation.

**Proposition 2.** *The cancellable drawdown insurance value admits the following decomposition:*

$$(22) \quad F(y, p) = f(y, p) + G(y, p),$$

where

$$(23) \quad G(y, p) := \sup_{\tau \in \mathcal{T}} g_\tau(y, p),$$

$$(24) \quad \tilde{f}(y, p) := -f(y, p) - c$$

for

$$(25) \quad g_\tau(y, p) := \mathbb{E}_{|y} \left[ e^{-r\tau} \tilde{f}(D_\tau, p); \tau < \tau_D^+(a) \right]$$

and  $f(\cdot, \cdot)$  is defined in (14).

*Proof.* Using  $\mathbb{1}_{(\tau \geq \tau_D^+(a))} = 1 - \mathbb{1}_{(\tau < \tau_D^+(a))}$  in (21) we obtain:

$$\begin{aligned} F(y, p) &= \mathbb{E}_{|y} \left[ - \int_0^{\tau_D^+(a)} e^{-rt} p dt + \alpha e^{-r\tau_D^+(a)} \right] \\ &\quad + \sup_{\tau \in \mathcal{T}} \mathbb{E}_{|y} \left[ \int_{\tau \wedge \tau_D^+(a)}^{\tau_D^+(a)} e^{-rt} p dt - \alpha e^{-r\tau_D^+(a)} \mathbb{1}_{(\tau < \tau_D^+(a))} - ce^{-r\tau} \mathbb{1}_{(\tau < \tau_D^+(a))} \right]. \end{aligned}$$

Note that the first term does not depend on  $\tau$ . The second term depends on  $\tau$  only through  $\tau < \tau_D^+(a)$ . Then by strong Markov property we get:

$$\begin{aligned} F(y, p) &= f(y, p) \\ &\quad + \sup_{\tau \in \mathcal{T}} \mathbb{E}_{|y} \left[ \int_\tau^{\tau_D^+(a)} e^{-rt} p dt - \alpha e^{-r\tau_D^+(a)} \mathbb{1}_{(\tau < \tau_D^+(a))} - ce^{-r\tau} \mathbb{1}_{(\tau < \tau_D^+(a))} \right] \\ &= f(y, p) + \sup_{\tau \in \mathcal{T}} \mathbb{E}_{|y} \left[ e^{-r\tau} \mathbb{E}_{|D_\tau} \left( \int_0^{\tau_D^+(a)} e^{-rt} p dt - \alpha e^{-r\tau_D^+(a)} - c \right); \tau < \tau_D^+(a) \right]. \end{aligned}$$

This completes the proof.  $\square$

Observe now that  $\tilde{f}(y, p)$  in (24) is a decreasing function with respect to  $y$ . Thus, if  $\tilde{f}(0+, p) < 0$ , then the optimal stopping strategy for the investor is to never terminate the contract, that is  $\tau = \infty$ . To eliminate this trivial case we assume from now that

$$(26) \quad \tilde{f}(0+, p) > 0$$

which is equivalent to saying that

$$(27) \quad \frac{p}{r} - c > \left( \frac{p}{r} + \alpha \right) \xi(0+) \geq 0.$$

In order to determine the optimal cancellation strategy for our contract it is sufficient to solve the optimal stopping problem represented by second term in (22), that is to identify  $G(y, p)$ . We will use the „guess and verify” approach. This means that we first guess the candidate stopping rule and then verify if this is truly the optimal stopping rule using the Verification Lemma given below.

**Lemma 3.1.** *Let  $\Upsilon_t$  be a right-continuous process living in some Borel state space  $\mathbb{B}$  killed at some  $\mathcal{F}_t^\Upsilon$ -stopping time  $\tau_0$ , where  $\mathcal{F}_t^\Upsilon$  is a right-continuous natural filtration of  $\Upsilon$ . Consider the following stopping problem:*

$$(28) \quad v(\phi) = \sup_{\tau \in \mathcal{T}^\Upsilon} \mathbb{E} [e^{-r\tau} V(\Upsilon_\tau) | \Upsilon_0 = \phi]$$

for some function  $V$  and the family  $\mathcal{F}_t^\Upsilon$ -stopping times  $\mathcal{T}^\Upsilon$ . Assume that

$$(29) \quad \mathbb{P}(\lim_{t \rightarrow \infty} e^{-rt} V(\Upsilon_t) < \infty | \Upsilon_0 = \phi) = 1.$$

The pair  $(v^*, \tau^*)$  is a solution of stopping problem (28), that is

$$v^*(\phi) := \mathbb{E} [e^{-r\tau^*} V(\Upsilon_{\tau^*}) | \Upsilon_0 = \phi],$$

if the following conditions are satisfied:

- i)  $v^*(\phi) \geq V(\phi)$  for all  $\phi \in \mathbb{B}$ ,
- ii) the process  $e^{-rt} v^*(\Upsilon_t)$  is right continuous supermartingale.

*Proof.* The proof follows the same arguments as the proof of [3, Lem. 9.1, p. 240]; see also [9, Th. 2.2, p. 29].  $\square$

Using above Verification Lemma 3.1 we will prove that the first passage time of the drawdown process  $D_t$  below some level  $\theta$  is the optimal stopping time for (23) (hence also for (22)). That is, we will prove that

$$(30) \quad \tau^* = \tau_D^-(\theta) \in \mathcal{T}$$

for an appropriate chosen  $\theta \in [0, a)$ .

For the stopping rule (30) and for  $y > \theta$  we will compute explicitly  $g_{\tau_D^-(\theta)}(y, p)$  given in (25). Note that, if  $y > \theta$ , then

$$(31) \quad g(y, p, \theta) := g_{\tau_D^-(\theta)}(y, p) = \tilde{f}(\theta, p) \mathbb{E}_{|y} [e^{-r\tau_D^-(\theta)}; \tau_D^-(\theta) < \tau_D^+(a)] = \tilde{f}(\theta, p) \frac{W^{(r)}(a - y)}{W^{(r)}(a - \theta)}.$$

Furthermore, if  $y \leq \theta$  then the investor will terminate contract immediately:

$$(32) \quad g(y, p, \theta) = \mathbb{E}_{|y} [e^{-r\tau_D^-(\theta)} \tilde{f}(D_{\tau_D^-(\theta)}, p); \tau_D^-(\theta) < \tau_D^+(a)] = \tilde{f}(y, p).$$

Thus, for  $\theta \in [0, y]$  we have:

$$(33) \quad \begin{aligned} F(y, p, \theta) &= f(y, p) + g(y, p, \theta) \\ &= \left(\frac{p}{r} + \alpha\right) Z^{(r)}(a - y) + \left(\frac{p}{r} - c\right) \frac{W^{(r)}(a - y)}{W^{(r)}(a - \theta)} - \left(\frac{p}{r} + \alpha\right) Z^{(r)}(a - \theta) \frac{W^{(r)}(a - y)}{W^{(r)}(a - \theta)} - \frac{p}{r}. \end{aligned}$$

Recall that by (22) the optimal level  $\theta^*$  maximizes both value functions  $F(y, p)$  and  $G(y, p)$ . Thus

$$(34) \quad \theta^* = \inf \left\{ \theta \in [0, a) : \frac{\partial}{\partial \theta} g(y, p, \theta) = 0 \quad \text{and for all } \varsigma \geq 0 \quad g(y, p, \varsigma) \leq g(y, p, \theta) \right\}.$$

Note that  $\theta^* > 0$  because  $g(y, p, \theta)$  increases at  $\theta = 0$ . Indeed,

$$\left[ \frac{\partial}{\partial \theta} g(y, p, \theta) \right]_{\theta=0} = \tilde{f}'(0) \frac{W^{(r)}(a - y)}{W^{(r)}(a)} + \tilde{f}(0) \frac{W^{(r)}(a - y) W'^{(r)}(a)}{(W^{(r)}(a))^2} > 0,$$

where the last inequality follows from assumption (26) and fact that  $\tilde{f}'(0) = -(\frac{p}{r} + \alpha)\xi'(0) = 0$ . We will verify now that (30) indeed holds true, that is,  $\tau_D^-(\theta^*)$  is an optimal stopping rule.

**Theorem 3.** *Assume that (27) holds. The stopping time  $\tau_D^-(\theta^*)$ , with  $\theta^*$  defined in (34), is the optimal stopping rule for the stopping problems (23) and (21). Moreover, the price of the drawdown insurance contract with cancellable feature equals  $F(y, p) = f(y, p) + g(y, p, \theta^*)$ .*

*Proof.* Based on the optimal stopping problem (23) it is sufficient to check that  $\tau^* = \tau_D^-(\theta^*)$  fulfills two conditions presented in Verification Lemma 3.1 with  $\Upsilon_t = D_t$ ,  $\mathbb{B} = \mathbb{R}_+$ ,  $\tau_0 = \tau_D^+(a)$ ,  $V(x) = \tilde{f}(x, p)$ . Note that in this case the assumption (29) is clearly satisfied. In order to prove (i) of Verification

Lemma 3.1 it suffices to show that  $g(y, p, \theta) - \tilde{f}(y, p) \geq 0$  for some  $\theta$ . Observe that taking  $\theta = y$  for  $y \in (0, a)$  produces:

$$(35) \quad g(y, p, a) - \tilde{f}(y, p) = \mathbb{E}_{|y} \left[ e^{-r\tau_D^-(y)} \tilde{f}(D_{\tau_D^-(y)}, p); \tau_D^-(y) < \tau_D^+(a) \right] - \tilde{f}(y, p) = 0,$$

where  $\tilde{f}(y, p)$  is given in (24). Thus condition (i) of Verification Lemma 3.1 follows from the fact that  $\theta^*$  maximizes the function  $g(y, p, \cdot)$ .

To prove condition (ii) of Verification Lemma 3.1 note that by the strong Markov property the process

$$e^{-r(t \wedge \tau_D^+(a) \wedge \tau_D^-(\theta^*))} g(D_{t \wedge \tau_D^+(a) \wedge \tau_D^-(\theta^*)}, p, \theta^*) = \mathbb{E}_{|y} [e^{-rs} g(D_s, p, \theta^*) | \mathcal{F}_{t \wedge \tau_D^+(a) \wedge \tau_D^-(\theta^*)}]$$

is a martingale. Hence

$$\mathcal{A}_D g(y, p, \theta^*) - r g(y, p, \theta^*) = 0$$

for  $y \in (\theta^*, a)$  and for the full generator  $\mathcal{A}_D$  of the process  $D$ . Moreover, from (32) we know that for  $y \in (0, \theta^*)$  we have  $g(y, p, \theta^*) = \tilde{f}(y, p)$ . Therefore, for  $y \in (0, \theta^*)$ ,

$$\begin{aligned} \mathcal{A}_D g(y, p, \theta^*) - r g(y, p, \theta^*) &= \mathcal{A}_D \tilde{f}(y, p) - r \tilde{f}(y, p) \\ &= -\mathcal{A}_D f(y, p) + r f(y, p) + r c \\ &= -\left(\frac{p}{r} + \alpha\right) [\mathcal{A}_D \xi(y) - r \xi(y)] - r \left(\frac{p}{r} - c\right). \end{aligned}$$

Now, the strong Markov property of the process  $D_t$  implies that process  $e^{-rt \wedge \tau_D^+(a)} \xi(D_{t \wedge \tau_D^+(a)}) = \mathbb{E}_{|y} [e^{-rs} \xi(D_s) | \mathcal{F}_{t \wedge \tau_D^+(a)}]$  is a martingale. Hence  $\mathcal{A}_D \xi(y) - r \xi(y) = 0$  for  $y \in (0, \theta^*)$  since  $\theta^* < a$ .

Thus the process  $e^{-rt \wedge \tau_D^+(a)} g(D_{t \wedge \tau_D^+(a)}, p, \theta^*)$  is a supermartingale because for  $y \in (0, \theta^*)$  we have:

$$\mathcal{A}_D g(y, p, \theta^*) - r g(y, p, \theta^*) = -r \left(\frac{p}{r} - c\right) \leq 0,$$

where the last inequality follows from the assumption (27). This completes the proof.  $\square$

#### 4. INCORPORATING DRAWUP COTINGENCY

**4.1. Fair premium.** The investor might like to buy a contract which has some maturity conditions. This means that this contract would end when these conditions are fulfilled. Therefore in this paper we also consider the insurance contract which provides protection from any specified drawdown of log-return of the asset price with certain drawup contingency. In particular, this contract may expire earlier if a fixed drawup event of log-return of the stock price occurs prior to some fixed drawdown of it. Choosing the drawup event is very natural since it corresponds to some market upward trend and therefore the investor may want to stop paying premium when this event happens. Under a risk-neutral measure the value of this contract equals:

$$(36) \quad k(y, z, p) := \mathbb{E}_{|y|z} \left[ - \int_0^{\tau_D^+(a) \wedge \tau_U^+(b)} e^{-rt} p dt + \alpha e^{-r\tau_D^+(a)} \mathbb{1}_{(\tau_D^+(a) \leq \tau_U^+(b))} \right],$$

for some fixed  $a > b > 0$ . At the beginning we will find this value function and then we identify the fair premium  $p^*$  under which

$$(37) \quad k(y, z, p^*) = 0.$$

Note that

$$(38) \quad k(y, z, p) = \left(\frac{p}{r} + \alpha\right) \nu(y, z) + \frac{p}{r} \lambda(y, z) - \frac{p}{r},$$

where

$$\begin{aligned} \nu(y, z) &:= \mathbb{E}_{|y|z} \left[ e^{-r\tau_D^+(a)}; \tau_D^+(a) \leq \tau_U^+(b) \right], \\ \lambda(y, z) &:= \mathbb{E}_{|y|z} \left[ e^{-r\tau_U^+(b)}; \tau_U^+(b) < \tau_D^+(a) \right]. \end{aligned}$$

To get the formulas for  $\nu$  and  $\lambda$  we have to make some additional observations.



**Proposition 3.** *Let  $y$  and  $z$  denote starting position for drawdown and drawup processes, respectively. For  $a > b \geq 0$  the following events are equivalent:*

$$(39) \quad \begin{aligned} & \left\{ \tau_U^+(b) < \tau_D^+(a), D_0 = y, U_0 = z \right\} = \left\{ \tau_{b-z}^+ < \tau_{(y-a) \vee (-z)}^- \right\} \\ & \cup \left\{ \overline{X}_{\tau_U^+(b)} \vee y - \underline{X}_{\tau_U^+(b)} < a, \underline{X}_{\tau_U^+(b)} \leq -z \right\}, \\ & \left\{ \tau_D^+(a) < \tau_U^+(b), D_0 = y, U_0 = z \right\} = \left\{ \tau_{y-a}^- < \tau_{(b-z)}^+, y - a \geq -z \right\} \end{aligned}$$

$$(40) \quad \cup \left\{ \overline{X}_{\tau_U^+(b)} \vee y - \underline{X}_{\tau_U^+(b)} \geq a, \underline{X}_{\tau_U^+(b)} \leq -z, \overline{X}_{\tau_U^+(b)} \leq b - z, y - a < -z \right\}.$$

*Proof.* The proof of this proposition follows from the geometric path arguments. To prove (39) note that

$$\begin{aligned} \left\{ \tau_U^+(b) < \tau_D^+(a), D_0 = y, U_0 = z \right\} &= \left\{ \tau_U^+(b) < \tau_D^+(a), \underline{X}_{\tau_U^+(b)} > -z, D_0 = y, U_0 = z \right\} \\ &\cup \left\{ \tau_U^+(b) < \tau_D^+(a), \underline{X}_{\tau_U^+(b)} \leq -z, D_0 = y \right\}. \end{aligned}$$

The event  $\{\underline{X}_{\tau_U^+(b)} > -z, U_0 = z\}$  is equivalent to the requirement that  $X_{\tau_U^+(b)} = b - z$  and that the underlying process  $X_t$  cannot cross  $y - a$  level before  $\tau_U^+(b)$ . Therefore,

$$(41) \quad \left\{ \tau_U^+(b) < \tau_D^+(a), \underline{X}_{\tau_U^+(b)} > -z, D_0 = y, U_0 = y \right\} = \left\{ \tau_{b-z}^+ < \tau_{(y-a) \vee (-z)}^- \right\}.$$

Moreover, when  $\underline{X}_{\tau_U^+(b)} \leq -z$  we have  $X_{\tau_U^+(b)} = b - \underline{X}_{\tau_U^+(b)}$ . Using now the fact that the drawdown occurs after drawup, we can conclude that  $D_{\tau_U^+(b)} < a - b$ . Thus

$$(42) \quad \left\{ \tau_U^+(b) < \tau_D^+(a), \underline{X}_{\tau_U^+(b)} \leq -z, D_0 = y \right\} = \left\{ \overline{X}_{\tau_U^+(b)} \vee y - \underline{X}_{\tau_U^+(b)} < a, \underline{X}_{\tau_U^+(b)} \leq -z \right\}.$$

Observations (41) and (42) complete the proof of (39). To prove (40) we again consider two scenarios:

$$\begin{aligned} \left\{ \tau_D^+(a) < \tau_U^+(b), D_0 = y, U_0 = z \right\} &= \left\{ \tau_D^+(a) < \tau_U^+(b), D_0 = y, U_0 = z, y - a < -z \right\} \\ &\cup \left\{ \tau_D^+(a) < \tau_U^+(b), D_0 = y, U_0 = z, y - a \geq -z \right\}. \end{aligned}$$

The case  $y - a > -z$  together with the assumption  $b \leq a$  imply that  $b - a < y$ . This means that the event when drawdown occurs before drawup is the same as the one when process  $X$  crosses  $y - a$  before it hits  $b - z$ . That is,

$$\left\{ \tau_D^+(a) < \tau_U^+(b), D_0 = y, U_0 = z, y - a \geq -z \right\} = \left\{ \tau_{y-a}^- < \tau_{(b-z)}^+, y - a \geq -z \right\}.$$

If  $y - a \leq -z$  then the process  $X$  is crossing  $-z$  before the drawdown event occurs. Additionally, the underlying process  $X$  can cross level  $y$  but it cannot cross level  $b - z$ , because otherwise the drawup would occur. Thus,

$$\begin{aligned} & \left\{ \tau_D^+(a) < \tau_U^+(b), D_0 = y, U_0 = z, y - a < -z \right\} \\ &= \left\{ \overline{X}_{\tau_U^+(b)} \vee y - \underline{X}_{\tau_U^+(b)} > a, \underline{X}_{\tau_U^+(b)} \leq -z, \underline{X}_{\tau_U^+(b)} \leq b - z, y - a < -z \right\}. \end{aligned}$$

This completes the proof of (40).  $\square$

Note that, for  $b < a$ , we have

$$\mathbb{E}_{|y|z} \left[ e^{-r\tau_U^+(b)}; \tau_D^+(a) < \tau_U^+(b) \right] = \mathbb{E}_{|y|z} \left[ e^{-r\tau_D^+(a)}; \tau_D^+(a) < \tau_U^+(b) \right] \mathbb{E} \left[ e^{-r\tau_U^+(b)} \right]$$

Proposition 3 and above observation produce the following crucial corollary.

**Corollary 4.** *For  $a > b$  we have:*

$$\begin{aligned} \nu(y, z) &= \mathbb{E} \left[ e^{-r\tau_{y-a}^-}; \tau_{y-a}^- < \tau_{(b-z)}^+ \right] \mathbb{1}_{(y+z \geq a)} \\ &+ \mathbb{E} \left[ e^{-r\tau_U^+(b)}; \overline{X}_{\tau_U^+(b)} \vee y - \underline{X}_{\tau_U^+(b)} \geq a, \underline{X}_{\tau_U^+(b)} \leq -z, \overline{X}_{\tau_U^+(b)} \leq b - z \right] \frac{\mathbb{1}_{(y+z < a)}}{\mathbb{E} \left[ e^{-r\tau_U^+(b)} \right]} \end{aligned}$$

and

$$\begin{aligned} \lambda(y, z) &= \mathbb{E} \left[ e^{-r\tau_{b-z}^+}; \tau_{b-z}^+ < \tau_{(y-a) \vee (-z)}^- \right] \\ &+ \mathbb{E} \left[ e^{-r\tau_U^+(b)}; \overline{X}_{\tau_U^+(b)} \vee y - \underline{X}_{\tau_U^+(b)} < a, \underline{X}_{\tau_U^+(b)} \leq -z \right]. \end{aligned}$$

Both functions  $\lambda$  and  $\nu$  can be now calculated by taking inverse Laplace transform of (12).

**Theorem 5.** *The price of the contract (36) is given in (38) with  $\lambda$  and  $\nu$  identified in Corollary 4.*

From (38) it follows the following theorem.

**Theorem 6.** *For the contract (36) the fair premium defined in (37) equals:*

$$(43) \quad p^* = \frac{r\alpha\nu(y, z)}{1 - \lambda(y, z) - \nu(y, z)},$$

where functions  $\lambda$  and  $\nu$  are given in Corollary 4.

**4.2. Cancellable feature.** We will also consider additional possibility of terminating the previous contract. Now, the protection buyer can terminate the position by paying fee  $c \geq 0$  for doing it. The value of this contract equals then

$$(44) \quad K(y, z, p) := \sup_{\tau \in \mathcal{T}} \mathbb{E}_{|y|z} \left[ - \int_0^{\tau_D^+(a) \wedge \tau_U^+(b) \wedge \tau} e^{-rt} p dt \right. \\ \left. + \alpha e^{-r\tau_D^+(a)} \mathbb{1}_{(\tau_D^+(a) < \tau_U^+(b) \wedge \tau)} - c e^{-r\tau} \mathbb{1}_{(\tau < \tau_D^+(a) \wedge \tau_U^+(b))} \right].$$

Similarly like in the case of cancellable drawdown contract we can represent the contract value function as the sum of two parts: one without cancellable feature and one that depends on a stopping time  $\tau$ .

**Proposition 4.** *The cancellable drawup insurance value admits the decomposition:*

$$(45) \quad K(y, z, p) = k(y, z, p) + H(y, z, p),$$

where

$$(46) \quad H(y, z, p) := \sup_{\tau \in \mathcal{T}} h_\tau(y, z, p),$$

$$(47) \quad h_\tau(y, z, p) := \mathbb{E}_{|y|z} \left[ e^{-r\tau} \tilde{k}(D_\tau, U_\tau, p); \tau < \tau_D^+(a) \wedge \tau_U^+(b) \right],$$

$$(48) \quad \tilde{k}(y, z, p) := -k(y, z, p) - c$$

and  $k$  is given in (38).

*Proof.* The proof follows from the following equality:

$$(49) \quad K(y, z, p) = \mathbb{E}_{|y|z} \left[ - \int_0^{\tau_D^+(a) \wedge \tau_U^+(b)} e^{-rt} p dt + \alpha e^{-r\tau_D^+(a)} \mathbb{1}_{(\tau_D^+(a) < \tau_U^+(b))} \right] \\ + \sup_{\tau \in \mathcal{T}} \mathbb{E}_{|y|z} \left[ e^{-r\tau} \mathbb{1}_{(\tau < \tau_D^+(a) \wedge \tau_U^+(b))} \mathbb{E}_{|D_\tau|U_\tau} \left[ \int_0^{\tau_D^+(a) \wedge \tau_U^+(b)} e^{-rt} p dt \right] \right. \\ \left. - \alpha e^{-r\tau_D^+(a)} \mathbb{1}_{(\tau < \tau_D^+(a) < \tau_U^+(b))} - c e^{-r\tau} \mathbb{1}_{(\tau < \tau_D^+(a) \wedge \tau_U^+(b))} \right].$$

□

At the beginning note that, if  $\tilde{k}(D_{\tau_D^-(\theta)}, U_{\tau_D^-(\theta)}) < 0$  for all  $\theta$ , then it is not optimal to terminate the contract and hence  $\tau = \infty$ . For avoiding this case we assume from this point, that there exist  $\theta_0$  for which  $\tilde{k}(D_{\tau_D^-(\theta_0)}, U_{\tau_D^-(\theta_0)}) > 0$ . We can rewrite this assumption as follows:

$$(50) \quad \frac{p}{r} - c > \left( \frac{p}{r} + \alpha \right) \nu(\theta_0, y + z - \theta_0) + \frac{p}{r} \lambda(\theta_0, y + z - \theta_0) \geq 0$$

for  $y + z \geq a$  and

$$(51) \quad \frac{p}{r} - c > \left( \frac{p}{r} + \alpha \right) \nu(\theta_0, y - x_0 - \theta_0) + \frac{p}{r} \lambda(\theta_0, y - x_0 - \theta_0) \geq 0$$

for  $y + z < a$ , where  $x_0$  satisfies  $\tilde{k}(\theta_0, y - x_0 - \theta_0) = \min_{x \in (y-a, -z)} \tilde{k}(\theta_0, y - x - \theta_0)$ . Additionally, because of the presence of indicator in (47) without loss of generality we can assume that

$$b - z > y - \theta.$$

To identify the value of the contract  $K$  we will find now the function  $H$  defined in (46). We will again use the „guess and verify” approach. The candidate for optimal strategy is given by

$$(52) \quad \tau^* = \tau_D^-(\theta)$$

for some  $\theta \in [0, a)$ . We denote:

$$(53) \quad h(y, z, p, \theta) := h_{\tau_D^-(\theta)}(y, z, p).$$

We will find this function now. Note that for  $\theta > y$  we have

$$(54) \quad h(y, z, p, \theta) = \tilde{k}(y, z, p).$$

Moreover,

$$U_{\tau_D^-(\theta)} = X_{\tau_D^-(\theta)} - \underline{X}_{\tau_D^-(\theta)} \wedge (-z) = y - \theta - \underline{X}_{\tau_D^-(\theta)} \wedge (-z).$$

Thus, by considering two disjoint possible scenarios  $\{\underline{X}_{\tau_D^-(\theta)} > -z\}$  and  $\{\underline{X}_{\tau_D^-(\theta)} \leq -z\}$ , the expectation in (47) can be rewritten for  $y \geq \theta$  as follows:

$$(55) \quad \begin{aligned} h(y, z, p, \theta) = & \tilde{k}(\theta, y - \theta + z, p) \mathbb{E}_{|y|z} \left[ e^{-r\tau_D^-(\theta)}; \tau_D^-(\theta) < \tau_D^+(a) \wedge \tau_U^+(b), \underline{X}_{\tau_D^-(\theta)} > -z \right] \\ & + \mathbb{E}_{|y|z} \left[ e^{-r\tau_D^-(\theta)} \tilde{k}(\theta, y - \theta - \underline{X}_{\tau_D^-(\theta)}, p); \tau_D^-(\theta) < \tau_D^+(a) \wedge \tau_U^+(b), \underline{X}_{\tau_D^-(\theta)} \leq -z \right]. \end{aligned}$$

We will analyze now the event appearing in last both increments.

**Proposition 5.** *The following events are equivalent:*

$$(56) \quad \{\tau_D^-(\theta) < \tau_D^+(a) \wedge \tau_U^+(b), D_0 = y, U_0 = z\} = \left\{ \underline{X}_{\tau_D^-(\theta)} > y - a, \underline{X}_{\tau_D^-(\theta)} \wedge (-z) > y - \theta - b \right\}.$$

*Proof.* Note that stopping time  $\tau_D^-(\theta)$  occurs when the process  $X$  hits  $y - \theta$ . This means that  $X$  cannot exceed level  $y$  before  $\tau_D^-(\theta)$  and therefore we have  $\bar{X}_{\tau_D^-(\theta)} \vee y = y$ . Now, the first event  $\{\underline{X}_{\tau_D^-(\theta)} > y - a\}$  on the right hand side of (56) corresponds to the situation when  $\tau_D^+(a)$  occurs after  $\tau_D^-(\theta)$ . On the second event  $\{\underline{X}_{\tau_D^-(\theta)} \wedge (-z) > y - \theta - b\}$  on the right hand side (56), the drawup process  $U$  attains level  $b$  only after the first passage time of  $y - \theta$  by the process  $X$ . In this case  $\tau_U^+(b)$  also cannot occur before  $\tau_D^-(\theta)$ . This observation completes the proof.  $\square$

Proposition 5 and (55) give the following representation of the function  $h$  defined formally in (53).

**Lemma 1.** *For  $y \geq \theta$  we have:*

$$(57) \quad \begin{aligned} h(y, z, p, \theta) = & \tilde{k}(\theta, y + z - \theta) \frac{W^{(r)}((a - y) \wedge z)}{W^{(r)}(y - \theta + (a - y) \wedge z)} \mathbb{1}_{(y - \theta < b - z)} \\ & + \mathbb{E}_{|y} \left[ e^{-r\tau_D^-(\theta)} \tilde{k}(\theta, y - \theta - \underline{X}_{\tau_D^-(\theta)}, p); (y - a) \vee (y - \theta - b) < \underline{X}_{\tau_D^-(\theta)} \leq -z \right] \mathbb{1}_{((y - a) \vee (y - \theta - b) < -z)}. \end{aligned}$$

Observe that in order to calculate (53) (or (57)) we only need to know the joint distribution of  $\underline{X}_{\tau_D^-(\theta)}$  and  $\tau_D^-(\theta)$ . This can be derived using (13).

In order to satisfy (52) we look for  $\theta$  that maximizes the function  $h(y, z, p, \theta)$ . We denote

$$(58) \quad \theta^* = \inf \left\{ \theta \in [0, a) : \frac{\partial}{\partial \theta} h(y, z, p, \theta) = 0 \quad \text{and for all } \varsigma \geq 0 \quad h(y, z, p, \varsigma) \leq h(y, z, p, \theta) \right\}.$$

Note that, if there is no local maximum of  $h$  on  $[0, a)$ , then  $\tau_D^-(\theta)$  is not the optimal stopping for considered problem.

**Theorem 7.** *Assume that (50) and (51) hold and there exists  $\theta^*$  defined in (58). Then  $\tau_D^-(\theta^*)$  with  $\theta^*$  given by (58) is the optimal stopping rule solution for (47) and the value of the contract (44) equals  $K(y, z, p) = k(y, z, p) + h(y, z, p, \theta^*)$  for  $h(y, z, p, \theta^*)$  given in (54) and (57) and  $k(y, z, p)$  given in (38).*

*Proof.* The optimization problem that we deal with here is defined in (46). We will use again Verification Lemma 3.1 for  $\Upsilon_t = (D_t, U_t)$ ,  $\mathbb{B} = \mathbb{R}_+ \times \mathbb{R}_+$ ,  $\tau_0 = \tau_U^+(b) \wedge \tau_D^+(a)$ ,  $V(\phi) = \tilde{k}(y, z, p)$  with  $\phi = (y, z)$ . The proof is similar to the proof of Theorem 3. The proof of the condition (i) of Verification Lemma 3.1 follows in fact the same idea with  $\theta = y$  at the first step.

To prove the second condition of Verification Lemma 3.1, note that, using strong Markov property for  $\Upsilon_t = (D_t, U_t)$ , we can conclude that the process

$$e^{-rt \wedge \tau_D^+(a) \wedge \tau_U^+(b) \wedge \tau_D^-(\theta^*)} h(D_{t \wedge \tau_D^+(a) \wedge \tau_U^+(b) \wedge \tau_D^-(\theta^*)}, U_{t \wedge \tau_D^+(a) \wedge \tau_U^+(b) \wedge \tau_D^-(\theta^*)}, p, \theta^*)$$

is a true martingale. Hence, for  $y > \theta^*$ ,

$$\mathcal{A}_{(D,U)}h(y, z, p, \theta^*) - rh(y, z, p, \theta^*) = 0,$$

where  $\mathcal{A}_{(D,U)}$  is a full generator of the process  $(D_t, U_t)$ . Moreover, the processes

$$e^{-rt \wedge \tau_D^+(a)} \nu(D_{t \wedge \tau_D^+(a)}, U_{t \wedge \tau_D^+(a)}), \quad e^{-rt \wedge \tau_U^+(b)} \lambda(D_{t \wedge \tau_U^+(b)}, U_{t \wedge \tau_U^+(b)})$$

are also  $\mathcal{F}_t$ -martingales. Thus  $\mathcal{A}_{(D,U)}\nu(y, z) - r\nu(y, z) = 0$  and  $\mathcal{A}_{(D,U)}\lambda(y, z) - r\lambda(y, z) = 0$ . Summing up, by (50), (51) and (54) for  $y \in (0, \theta^*)$  and  $z \in (0, b)$  we have:

$$\mathcal{A}_{(D,U)}h(y, z, p, \theta^*) - rh(y, z, p, \theta^*) = \mathcal{A}_{(D,U)}\tilde{k}(y, z, p) - r\tilde{k}(y, z, p) = -r \left( \frac{p}{r} - c \right) \leq 0.$$

This completes the proof.  $\square$

**4.3. Striking simple case when  $a = b$ .** The case where  $a = b$  corresponds to situation when contract pays the compensation when drawdown process exceed level  $a$  or expires when drawup obtains the same level  $a$ . In this subsection we will find function  $\lambda(y, z)$  and  $\nu(y, z)$  appearing in (38) and (43) (hence also in (45) and (57)).

The simplicity of this case follows from the fact that we can divide problem into two easy subcases. The first one is when  $a \leq z + y$ .

Then  $\lambda$  and  $\nu$  can be identified using the two-sided exit formulas (6)-(7). Indeed,

$$\begin{aligned} \lambda(y, z) &= \mathbb{E}_{|y|z} \left[ e^{-r\tau_U^+(a)}; \tau_U^+(a) < \tau_D^+(a) \right] = \mathbb{E} \left[ e^{-r\tau_{a-z}^+}; \tau_{a-z}^+ < \tau_{y-a}^- \right] \\ (59) \quad &= \frac{W^{(r)}(a - y)}{W^{(r)}(2a - y - z)} \end{aligned}$$

and

$$\begin{aligned} \nu(y, z) &= \mathbb{E}_{|y|z} \left[ e^{-r\tau_D^+(a)}; \tau_D^+(a) < \tau_U^+(a) \right] = \mathbb{E} \left[ e^{-r\tau_{y-a}^-}; \tau_{y-a}^- < \tau_{a-z}^+ \right] \\ (60) \quad &= Z^{(r)}(a - y) - Z^{(r)}(2a - y - z) \frac{W^{(r)}(a - y)}{W^{(r)}(2a - y - z)}. \end{aligned}$$

The second case is when  $a > z + y$ .

For this case the following identity is crucial:

$$(61) \quad \{ \tau_U^+(a) \in dt, \tau_D^+(a) > t, X_t \in dx \} = \{ \tau_x^+ \in dt, \underline{X}_t \wedge (-z) \in d(x - a) \}$$

that holds for any  $x \in (y, a - z]$ ; see [15, Eq. (46)]. Using (61) we observe that:

$$\begin{aligned} \lambda(y, z) &= \mathbb{E}_{|y|z} \left[ e^{-r\tau_U^+(a)}; \tau_U^+(a) < \tau_D^+(a) \right] \\ &= \int_0^\infty \int_0^\infty e^{-rt} \mathbb{P}_{|y|z} (\tau_U^+(a) \in dt; \tau_D^+(a) > t; X_t \in dx) \\ &= \int_0^\infty \int_0^\infty e^{-rt} \mathbb{P} (\tau_x^+ \in dt; \underline{X}_{\tau_x^+} \wedge (-z) \in d(x - a), x \in (y, a - z)) \\ &= \int_y^{a-z} \mathbb{E} \left[ e^{-r\tau_x^+}; \underline{X}_{\tau_x^+} \wedge (-z) \in dx - a \right] \\ &= \int_y^{a-z} \mathbb{E} \left[ e^{-r\tau_x^+}; \underline{X}_{\tau_x^+} \in d(x - a) \right] + \mathbb{E} \left[ e^{-r\tau_{a-z}^+}; \underline{X}_{\tau_{a-z}^+} > -z \right] \\ &= \int_y^{a-z} \frac{\partial}{\partial a} \frac{W^{(r)}(a - x)}{W^{(r)}(a)} dx + \mathbb{E} \left[ e^{-r\tau_{a-z}^+}; \underline{X}_{\tau_{a-z}^+} > -z \right] \\ (62) \quad &= \frac{W^{(r)}(a - y)}{W^{(r)}(a)} - \frac{1}{r} \frac{W^{(r)}(a)}{(W^{(r)}(a))^2} \left( Z^{(r)}(a - y) - Z^{(r)}(z) \right), \end{aligned}$$

where we use the fact that

$$(63) \quad \mathbb{E} \left[ e^{-r\tau_x^+}; \underline{X}_{\tau_x^+} \in -du \right] = \frac{\partial}{\partial u} \frac{W^{(r)}(u)}{W^{(r)}(x + u)} (-du).$$

The function  $\nu$  can be calculated using the formula for the Laplace transform  $\xi$  of  $\tau_D^+(a)$  given in (17) and above expression for  $\lambda$ . Indeed,

$$\begin{aligned}
\nu(y, z) &= \mathbb{E}_{|y|z} \left[ e^{-r\tau_D^+(a)}; \tau_D^+(a) < \tau_U^+(a) \right] \\
&= \mathbb{E}_{|y|z} \left[ e^{-r\tau_D^+(a)} \right] - \mathbb{E}_{|y|z} \left[ e^{-r\tau_D^+(a)}; \tau_U^+(a) < \tau_D^+(a) \right] \\
&= \mathbb{E}_{|y|} \left[ e^{-r\tau_D^+(a)} \right] - \mathbb{E}_{|y|z} \left[ e^{-r\tau_D^+(a)}; \tau_U^+(a) < \tau_D^+(a) \right] \mathbb{E} \left[ e^{-r\tau_D^+(a)} \right] \\
&= Z^{(r)}(a - y) - rW^{(r)}(a - y) \frac{W^{(r)}(a)}{W^{(r)}(a)} - \lambda(y, z) \left( Z^{(r)}(a) - rW^{(r)}(a) \frac{W^{(r)}(a)}{W^{(r)}(a)} \right) \\
(64) \quad &= Z^{(r)}(z) - Z^{(r)}(a) \frac{W^{(r)}(a - y)}{W^{(r)}(a)} + \frac{1}{r} Z^{(r)}(a) \frac{W^{(r)}(a)}{(W^{(r)}(a))^2} \left( Z^{(r)}(a - y) - Z^{(r)}(z) \right).
\end{aligned}$$

Identification of  $\nu$  and  $\lambda$  allows also to calculate the function  $h$  appearing in the value function (44) given in Theorem 7. Precisely, by (54) for  $\theta \leq y$  we have

$$h(y, z, p, \theta) = \tilde{k}(y, z, p),$$

where  $\tilde{k}$  could be identified using (38) and (48). For  $\theta > y$  we have

$$\begin{aligned}
h(y, z, p, \theta) &= \mathbb{E}_{|y} \left[ e^{-r\tau_D^-(\theta)} \tilde{k}(\theta, y - \theta - \underline{X}_{\tau_D^-(\theta)}); y - a < \underline{X}_{\tau_D^-(\theta)} < -z \right] \\
&\quad + \tilde{k}(\theta, y + z - \theta) \mathbb{E}_{|y} \left[ e^{-r\tau_D^-(\theta)}; \underline{X}_{\tau_D^-(\theta)} > (y - a) \vee (-z) \right] \mathbb{1}_{(a > y + z - \theta)} \\
&= \int_z^{a-y} \tilde{k}(\theta, y - \theta + \phi) \frac{\partial}{\partial \phi} \frac{W^{(r)}(\phi)}{W^{(r)}(y - \theta + \phi)} d\phi \mathbb{1}_{(a > y + z)} \\
&\quad + \tilde{k}(\theta, y + z - \theta) \frac{W^{(r)}((a - y) \wedge z)}{W^{(r)}((a - y) \wedge z + y - \theta)} \mathbb{1}_{(a > y + z - \theta)}.
\end{aligned}$$

Now, by integration-by-parts formula we have:

$$\begin{aligned}
h(y, z, p, \theta) &= \tilde{k}(\theta, a - \theta) \frac{W^{(r)}(a - y)}{W^{(r)}(a - \theta)} \mathbb{1}_{(a > y + z)} - \tilde{k}(\theta, y + z - \theta) \frac{W^{(r)}(z)}{W^{(r)}(y + z - \theta)} \mathbb{1}_{(a > y + z)} \\
&\quad + \tilde{k}(\theta, y + z - \theta) \frac{W^{(r)}((a - y) \wedge z)}{W^{(r)}((a - y) \wedge z + y - \theta)} \mathbb{1}_{(a > y + z - \theta)} \\
&\quad - \int_z^{a-y} \left[ -\left(\frac{p}{r} + \alpha\right) \left(1 - \frac{1}{r} Z^{(r)} \frac{W^{(r)}(a)}{(W^{(r)}(a))^2}\right) - \frac{p}{r} \frac{1}{r} \frac{W^{(r)}(a)}{(W^{(r)}(a))^2} \right] rW^{(r)}(\phi) d\phi \mathbb{1}_{(a > y + z)} \\
(65) \quad &= \begin{cases} \left[ -\left(\frac{p}{r} + \alpha\right) \left(1 - \frac{1}{r} Z^{(r)} \frac{W^{(r)}(a)}{(W^{(r)}(a))^2}\right) - \frac{p}{r} \frac{1}{r} \frac{W^{(r)}(a)}{(W^{(r)}(a))^2} \right] (Z^{(r)}(a - y) - Z^{(r)}(z)) \\ \quad + \tilde{k}(\theta, a - \theta) \frac{W^{(r)}(a - y)}{W^{(r)}(a - \theta)} & \text{for } a > y + z, \\ \tilde{k}(\theta, y + z - \theta) \frac{W^{(r)}(a - y)}{W^{(r)}(a - \theta)} & \text{for } y + z - \theta < a < y + z, \\ 0 & \text{for } a < y + z - \theta. \end{cases}
\end{aligned}$$

## 5. NUMERICAL ANALYSIS

In this section we analyze numerically all insurance contracts and check their dependence on chosen parameters of the model. We focus on two classical spectrally negative risk Lévy processes. The first one is the linear Brownian motion:

$$(66) \quad X_t = \mu t + \sigma B_t,$$

where  $B_t$  is a standard Brownian motion. The second process we analyze here is the classical Cramér-Lundberg process with the exponential jumps:

$$(67) \quad X_t = \mu t - \sum_{i=1}^{N_t} \eta_i,$$

where  $\eta_i$  are i.i.d. exponentially distributed random variables with the parameter  $\rho > 0$  and  $N_t$  is an independent Poisson process with intensity  $\beta > 0$ .

We express all main quantities and all contract values in terms of the scale functions defined in (3) and (4). From [4] it follows that the scale functions for the Brownian motion with drift (66) take the following form:

$$(68) \quad W^{(r)}(u) = \frac{2}{\sigma^2 \Xi} e^{-\frac{\mu}{\sigma^2} u} \sinh(\Xi u),$$

$$(69) \quad Z^{(r)}(u) = e^{-\frac{\mu}{\sigma^2} u} \left( \cosh(\Xi u) + \frac{\mu}{\Xi \sigma^2} \sinh(\Xi u) \right),$$

where

$$\Xi = \frac{\sqrt{\mu^2 + 2r\sigma^2}}{\sigma^2}.$$

Similarly, for the Cramér-Lundberg process (67),

$$(70) \quad W^{(r)}(u) = \frac{e^{\Phi(r)u}}{\psi'(\Phi(r))} + \frac{e^{\zeta u}}{\psi'(\zeta)},$$

$$(71) \quad Z^{(r)}(u) = 1 + r \frac{e^{\Phi(r)u} - 1}{\Phi(r)\psi'(\Phi(r))} + r \frac{e^{\zeta u} - 1}{\zeta\psi'(\zeta)},$$

where

$$\begin{aligned} \Phi(r) &= \frac{1}{2\mu} \left( (\beta + r - \mu\rho) + \sqrt{(\beta + r - \mu\rho)^2 + 4r\mu\rho} \right), \\ \zeta &= \frac{1}{2\mu} \left( (\beta + r - \mu\rho) - \sqrt{(\beta + r - \mu\rho)^2 + 4r\mu\rho} \right), \\ \psi'(\phi) &= \mu - \frac{\beta\rho}{(\rho + \phi)^2} \end{aligned}$$

and  $\psi(\cdot)$  is the Laplace exponent given in (1).

In this section we analyze the influence of chosen parameters of our model on the prices and the stopping rules of considered insurance contracts. To simplify above comparison, we order the numerical analysis according to the order of the appearance of these contracts in this paper.

**5.1. Fair premium for drawdown insurance.** We start from pricing the contract (14) using (15). Let  $X_t$  be a linear Brownian motion (66). From Proposition 1 we have:

$$\xi(y) = e^{-\frac{\mu}{\sigma^2}(a-y)} \frac{\Xi \cosh(\Xi y) - \frac{\mu}{\sigma^2} \sinh(\Xi y)}{\Xi \cosh(\Xi a) - \frac{\mu}{\sigma^2} \sinh(\Xi a)}.$$

This leads to formula for the value function  $f(y, p)$  given in (14) and the expression for the fair premium  $p^*$  given in (20):

$$\begin{aligned} f(y, p) &= \left( \frac{p}{r} + \alpha \right) e^{-\frac{\mu}{\sigma^2}(a-y)} \frac{\Xi \cosh(\Xi y) - \frac{\mu}{\sigma^2} \sinh(\Xi y)}{\Xi \cosh(\Xi a) - \frac{\mu}{\sigma^2} \sinh(\Xi a)} - \frac{p}{r}, \\ p^* &= \frac{r\alpha(\Xi \cosh(\Xi y) - \frac{\mu}{\sigma^2} \sinh(\Xi y))}{\Xi(\cosh(\Xi a) - \cosh(\Xi y)) - \frac{\mu}{\sigma^2}(\sinh(\Xi a) - \sinh(\Xi y))}. \end{aligned}$$

In Figure 1 we depicted the fair premium  $p^*$  depending on the starting position of drawdown  $D_0 = y$ . Similar calculations are done for the Cramér-Lundberg process given in (67). In particular, we have:

$$\begin{aligned} \xi(y) &= 1 + r \frac{e^{\Phi(r)(a-y)} - 1}{\Phi(r)\psi'(\Phi(r))} + r \frac{e^{\zeta(a-y)} - 1}{\zeta\psi'(\zeta)} - r \left( \frac{e^{\Phi(r)(a-y)}}{\psi'(\Phi(r))} + \frac{e^{\zeta(a-y)}}{\psi'(\zeta)} \right) \\ &\quad \cdot \frac{\psi'(\zeta)e^{\Phi(r)a} + \psi'(\Phi(r))e^{\zeta a}}{\Phi(r)\psi'(\zeta)e^{\Phi(r)a} + \zeta\psi'(\Phi(r))e^{\zeta a}} =: c_0 + c_{\Phi(r)}e^{\Phi(r)(r)(a-y)} + c_{-\zeta}e^{-\zeta(a-y)}, \end{aligned}$$

where

$$\begin{aligned} c_0 &= 1 - \frac{r}{\Phi(r)\psi'(\Phi(r))} - \frac{r}{\zeta\psi'(\zeta)}, \\ c_{\Phi(r)} &= \frac{r}{\Phi(r)\psi'(\Phi(r))} - \frac{r}{\psi'(\Phi(r))} \frac{\psi'(\zeta)e^{\Phi(r)a} + \psi'(\Phi(r))e^{\zeta a}}{\Phi(r)\psi'(\zeta)e^{\Phi(r)a} + \zeta\psi'(\Phi(r))e^{\zeta a}}, \\ c_{-\zeta} &= \frac{r}{\zeta\psi'(\zeta)} - \frac{r}{\psi'(\zeta)} \frac{\psi'(\zeta)e^{\Phi(r)a} + \psi'(\Phi(r))e^{\zeta a}}{\Phi(r)\psi'(\zeta)e^{\Phi(r)a} + \zeta\psi'(\Phi(r))e^{\zeta a}}. \end{aligned}$$

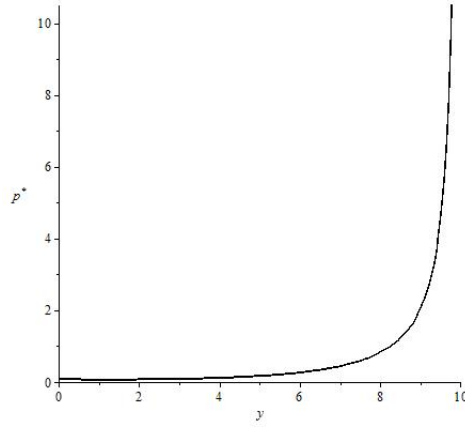


FIGURE 1. The value  $p^*$  for drawdown insurance contract for Brownian motion with drift. Parameters:  $r = 0.01, \mu = 0.03, \sigma = 0.4, \alpha = 100, a = 10$ .

The contract value  $f(y, p)$  given in (14) and the fair premium  $p^*$  given in (20) equal to

$$f(y, p) = \left( \frac{p}{r} + \alpha \right) \left( c_0 + c_{\Phi(r)} e^{\Phi(r)(a-y)} + c_{\zeta} e^{\zeta(a-y)} \right) - \frac{p}{r},$$

$$p^* = \frac{r\alpha(c_0 + c_{\Phi(r)} e^{\Phi(r)(a-y)} + c_{\zeta} e^{\zeta(a-y)})}{1 - c_0 - c_{\Phi(r)} e^{\Phi(r)(a-y)} - c_{\zeta} e^{\zeta(a-y)}}.$$

Figure 2 describes the dependence of the fair premium  $p^*$  on the starting drawdown  $D_0 = y$  in this case.

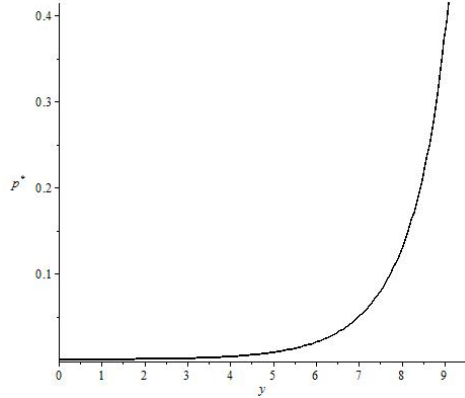


FIGURE 2. The value  $p^*$  for drawdown insurance contract for Cramér-Lundberg model. Parameters:  $r = 0.01, \mu = 0.05, \beta = 0.1, \rho = 2.5, \alpha = 100, a = 10$ .

Note that for both processes, the linear Brownian motion (66) and the Cramér-Lundberg model (67), both shapes of the fair premium are quite similar. Moreover, increasing starting drawdown  $D_0 = y$  may rapidly increase the value of the fair premium  $p^*$ . In fact, for the Brownian motion with drift, the value of  $p^*$  tends to  $\infty$  when  $y \uparrow a$ . This follows from the fact that for the linear Brownian motion we have  $W^{(r)}(0) = 0$  and the denominator in the expression for the fair premium  $p^*$  goes to 0 as  $y \uparrow a$ . For the Cramér-Lundberg process (67) this is not the case though. Indeed, for this process of bounded variation we have  $W^{(r)}(0) > 0$  and  $\lim_{y \rightarrow a^-} \xi(y) \neq 1$ , so the denominator in formula for fair premiums  $p^*$  does not converge to 0 as  $y \uparrow a$ .

**5.2. Cancellable drawdown insurance.** We now price the contract  $F(y, p) = f(y, p) + g(y, p, \theta^*)$  defined in (21) and identified in Theorem 3. We also find the optimal stopping rule  $\tau^*$  given in (30) for  $\theta^*$  defined in (34). Because of the numerical results presented in the previous subsection, it suffices to find only function  $g(y, p, \theta^*)$  and  $\theta^*$ .

For the linear Brownian motion model (66) we can explicitly write formula for the function  $g$ . In particular, for  $\theta < y$

$$g(y, p, \theta) = \tilde{f}(\theta, p) \frac{W^{(r)}(a - y)}{W^{(r)}(a - \theta)} = \left( \frac{p}{r} - c \right) \frac{e^{-\frac{\mu}{\sigma^2}(a-y)} \sinh(\Xi(a - y))}{e^{-\frac{\mu}{\sigma^2}(a-\theta)} \sinh(\Xi(a - \theta))} - \left( \frac{p}{r} + \alpha \right) \frac{e^{-\frac{\mu}{\sigma^2}(a-y)} \sinh(\Xi(a - y)) (\Xi \cosh(\Xi\theta) - \frac{\mu}{\sigma^2} \sinh(\Xi\theta))}{\sinh(\Xi(a - \theta)) (\Xi \cosh(\Xi\theta) - \frac{\mu}{\sigma^2} \sinh(\Xi\theta))}$$

and for  $\theta \geq y$  we have

$$(72) \quad g(y, p, \theta) = \tilde{f}(y, p) = - \left( \frac{p}{r} + \alpha \right) e^{-\frac{\mu}{\sigma^2}(a-y)} \frac{\Xi \cosh(\Xi y) - \frac{\mu}{\sigma^2} \sinh(\Xi y)}{\Xi \cosh(\Xi a) - \frac{\mu}{\sigma^2} \sinh(\Xi a)} + \frac{p}{r} - c.$$

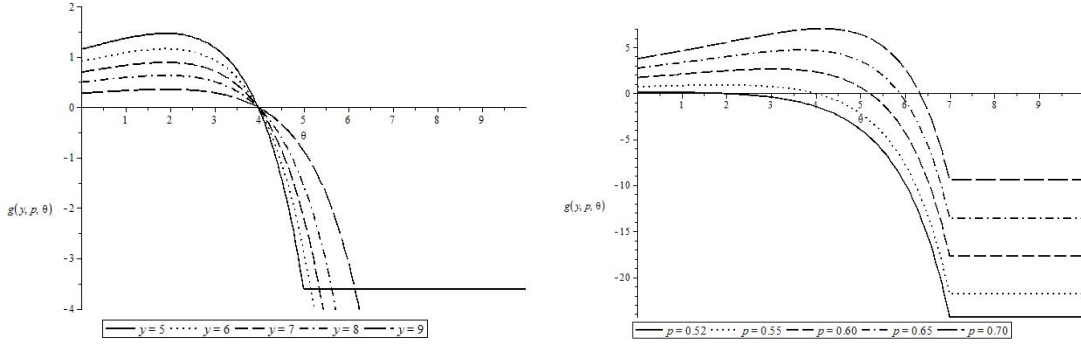


FIGURE 3. The value  $g(y, p, \theta)$  for Brownian motion with drift with respect to the  $\theta$  for different levels of  $y$  (first) and  $p$  (second). Parameters:  $r = 0.01, \mu = 0.03, \sigma = 0.4, \alpha = 100, c = 50, a = 10, p = 0.55, y = 7$ .

Figure 3 depicts the dependence of the function  $g(y, p, \theta)$  on  $\theta$ . The straight line at the end of the  $g(y, p, \theta)$  follows from the fact that by (32) the function  $g$  is constant equal to  $\tilde{f}(y, p)$  for  $\theta \geq y$ . We are looking for  $\theta^*$  that maximizes function  $g$ . When  $\theta^*$  is at the beginning of the straight line it means that the investor should not take this insurance contract. This is an extreme case. In fact, the most natural are the contract when  $\theta^*$  is between zero and beginning the straight line. Note also that by (31) and definition of the optimal  $\theta^*$  given in (34) the level  $\theta^*$  does not depend on the starting position of the drawdown  $D_0 = y$  as long as  $y > \theta^*$ . In particular, for our set of parameters, for constant premium, we have  $\theta^* \approx 2$ .

Figure 3 shows also the dependence of the function  $g(y, p, \theta)$  on the premium rate  $p$ . Note that higher premium rate produces higher values of function  $g$  hence also higher values of the insurance contract  $F$  given in (21).

For the Cramér–Lundberg model (67) the function  $g$  takes the following form:

$$g(y, p, \theta) = \tilde{f}(\theta, p) \frac{W^{(r)}(a - y)}{W^{(r)}(a - \theta)} = \frac{\psi'(\zeta) e^{\Phi(r)(a-y)} + \psi'(\Phi(r)) e^{\zeta(a-y)}}{\psi'(\zeta) e^{\Phi(r)(a-\theta)} + \psi'(\Phi(r)) e^{\zeta(a-\theta)}} \cdot \left[ - \left( \frac{p}{r} + \alpha \right) \left( c_0 + c_{\Phi(r)} e^{\Phi(r)(a-\theta)} + c_{\zeta} e^{\zeta(a-\theta)} \right) + \left( \frac{p}{r} - c \right) \right]$$

for  $\theta < y$  and

$$g(y, p, \theta) = \tilde{f}(y, p) = - \left( \frac{p}{r} + \alpha \right) \left( c_0 + c_{\Phi(r)} e^{\Phi(r)(a-y)} + c_{\zeta} e^{\zeta(a-y)} \right) + \frac{p}{r} - c$$

for  $\theta \geq y$ .

Figure 4 describes behaviour of the function  $g(y, p, \theta)$  on the stopping level  $\theta$ , hence identifying also the optimal one.

**5.3. Fair premium for drawup contingency when  $a > b$ .** We will now investigate numerically the insurance contract (36) which provides protection from any specified drawdown of size  $a$  with certain drawup contingency of size  $b$ . By Theorem 5 it suffices to find functions  $\lambda$  and  $\nu$  in order to identify the contract. As mentioned in Chapter 4, to do so, we will calculate these function using Corollary 4 and numerically inverting Laplace transform (12).



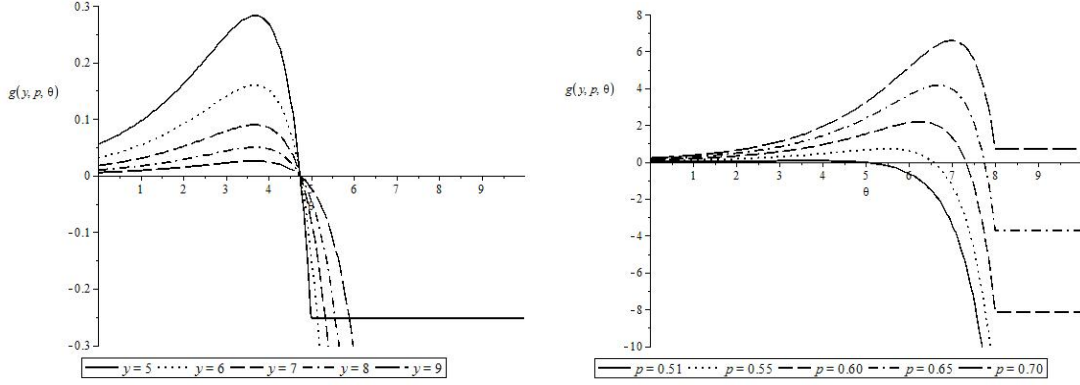


FIGURE 4. The value  $g(y, p, \theta)$  for Cramér-Lundberg model with respect to the  $\theta$  value for different levels of  $y$  (first) and  $p$  (second). Parameters:  $r = 0.01, \mu = 0.05, \beta = 0.1, \rho = 2.5, a = 10, \alpha = 100, c = 50, p = 0.51, y = 8$ .

It is worth to mention that in the case of linear Brownian motion (66) there exists alternative (to inverting Laplace transform) method of identifying  $\lambda$  and  $\nu$ . From Corollary 4 the formulas for  $\nu$  and  $\lambda$  reduce to two-sided formulas (6)-(7). That is, for  $a \leq y + z$ ,

$$\nu(y, z) = Z^{(r)}(a - y) - Z^{(r)}(a + b - y - z) \frac{W^{(r)}(a - y)}{W^{(r)}(a + b - y - z)}$$

and

$$\lambda(y, z) = \frac{W^{(r)}(a - y)}{W^{(r)}(a + b - y - z)}.$$

In order to identify  $\lambda$  and  $\nu$  when  $a > y + z$ , one can observe that process  $\hat{X}_t := -X_t$  is a linear Brownian motion with drift  $-\mu$ . Then we also have that  $\hat{U}_t = D_t$  and  $\hat{D}_t = U_t$ . By changing process  $X$  by  $\hat{X}$  we can use the Laplace transform of  $\hat{D}$  given in [6] and we can calculate the exact formulas for  $\nu$  and  $\lambda$ . Precisely, let  $\hat{W}^{(r)}$  and  $\hat{Z}^{(r)}$  be the scale functions for  $\hat{X}$  defined in (3) and (4). Then, for  $a > y + z$ ,

$$\begin{aligned} \nu(y, z) &= \frac{\hat{W}^{(r)}(b)}{\hat{W}'^{(r)}(b)} \frac{\sigma^2}{2} \left[ \frac{(\hat{W}'^{(r)}(b))^2}{\hat{W}^{(r)}(b)} - \hat{W}'^{(r)}(b) \right] e^{-(a-b\vee(y+z)) \frac{\hat{W}'^{(r)}(b)}{\hat{W}^{(r)}(b)}} Z^{(r)}(b) \\ &\quad - \frac{1}{r} \frac{1}{\hat{W}^{(r)}(b)} \frac{\sigma^2}{2} \left[ \frac{(\hat{W}'^{(r)}(b))^2}{\hat{W}^{(r)}(b)} - \hat{W}'^{(r)}(b) \right] \left( \hat{Z}^{(r)}(b - z) - \hat{Z}^{(r)}(y) \right) e^{-(a-b) \frac{\hat{W}'^{(r)}(b)}{\hat{W}^{(r)}(b)}} Z^{(r)}(b) \mathbb{1}_{(b > y+z)} \end{aligned}$$

and

$$\begin{aligned} \lambda(y, z) &= \frac{1}{r} \frac{1}{\hat{W}^{(r)}(b)} \frac{\sigma^2}{2} \left[ \frac{(\hat{W}'^{(r)}(b))^2}{\hat{W}^{(r)}(b)} - \hat{W}'^{(r)}(b) \right] \left( \hat{Z}^{(r)}(b \wedge (a - z)) - \hat{Z}^{(r)}(y) \right) e^{-(a-b) \frac{\hat{W}'^{(r)}(b)}{\hat{W}^{(r)}(b)}} \mathbb{1}_{(b > y)} \\ &\quad + \frac{\hat{W}^{(r)}(b)}{\hat{W}'^{(r)}(b)} \frac{\sigma^2}{2} \left[ \frac{(\hat{W}'^{(r)}(b))^2}{\hat{W}^{(r)}(b)} - \hat{W}'^{(r)}(b) \right] \left( e^{-z \frac{\hat{W}'^{(r)}(b)}{\hat{W}^{(r)}(b)}} - e^{-(a-b\vee y) \frac{\hat{W}'^{(r)}(b)}{\hat{W}^{(r)}(b)}} \right) \mathbb{1}_{(a > z+b)} + \frac{W^{(r)}(z)}{W^{(r)}(b)}. \end{aligned}$$

Using above expressions we can find the fair premium  $p^*$  defined in (43):

$$p^* = \frac{r\alpha\nu(y, z)}{1 - \lambda(y, z) - \nu(y, z)}$$

and analyze the dependence of starting position of drawdown and drawup processes on the fair premium  $p^*$ . Figure 5 shows this relation.

From Figure 5 we can deduce the same observation like we did for the basic drawdown contract. For the linear Brownian motion (66) the value of the fair premium  $p^*$  tends to  $\infty$  as  $y \uparrow a$ . This is because we have  $\lim_{y \rightarrow a^-} \nu(y, z) = 1$  and  $\lim_{y \rightarrow a^-} \lambda(y, z) = 0$ , and then, the denominator in the formula for  $p^*$  given in (43) converges to 0.

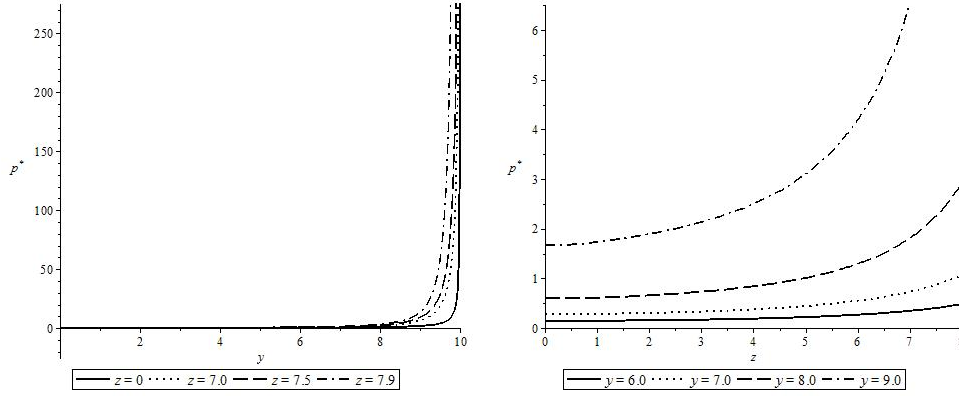


FIGURE 5. The value  $p^*$  for drawup contingency contract for Brownian motion with drift for different starting position of drawup process  $z$  (first) and different starting position of drawdown  $y$  (second). Parameters:  $r = 0.01, \mu = 0.03, \sigma = 0.4, \alpha = 100, a = 10, b = 8$ .

**5.4. Fair premium for drawup contingency when  $a = b$ .** We analyze here the special case when  $a = b$  presented in ubSection 4.3. This time we use identities (59), (60), (62) and (64) to compute the contract value.

Therefore, by the expression of the fair premium  $p^*$  given in (43), we can find that

$$(73) \quad p^* = \frac{r\alpha\nu(y, z)}{1 - \lambda(y, z) - \nu(y, z)} = \frac{r\alpha \left( Z^{(r)}(a - y) - Z^{(r)}(2a - y - z) \frac{W^{(r)}(a - y)}{W^{(r)}(2a - y - z)} \right)}{1 - Z^{(r)}(a - y) + \frac{W^{(r)}(a - y)}{W^{(r)}(2a - y - z)} (Z^{(r)}(2a - y - z) - 1)}$$

for  $a \leq y + z$  and

$$(74) \quad p^* = \frac{r\alpha \left( Z^{(r)}(z) - Z^{(r)}(a) \frac{W^{(r)}(a - y)}{W^{(r)}(a)} + \frac{1}{r} Z^{(r)}(a) \frac{W'^{(r)}(a)}{(W^{(r)}(a))^2} (Z^{(r)}(a - y) - Z^{(r)}(z)) \right)}{1 - Z^{(r)}(z) - (Z^{(r)}(a) - 1) \left( \frac{1}{r} \frac{W'^{(r)}(a)}{(W^{(r)}(a))^2} (Z^{(r)}(a - y) - Z^{(r)}(z)) - \frac{W^{(r)}(a - y)}{W^{(r)}(a)} \right)}.$$

for  $a > y + z$ . Using the formulas (70)-(71) for the scale functions for Cramér–Lundberg model (67), one can find the dependence of the fair premium  $p^*$  on the initial starting/historical positions of the drawdown  $D_0 = y$  and drawup  $U_0 = z$ . These dependence is depicted in Figure 6. Note that, similarly

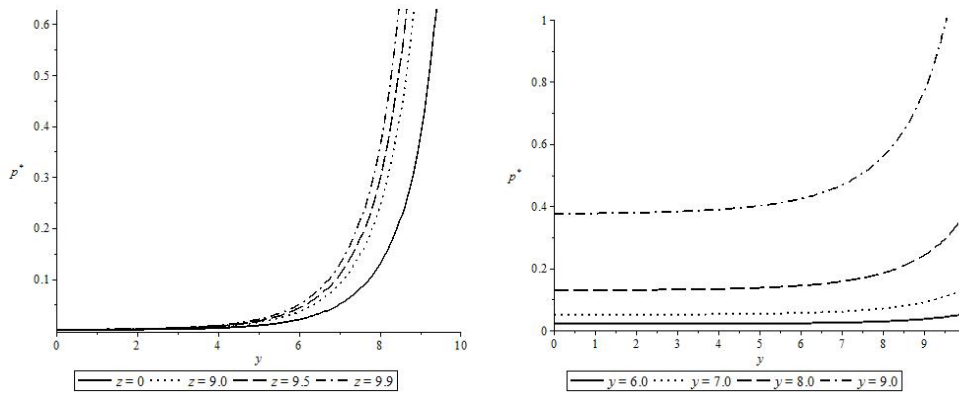


FIGURE 6. The value  $p^*$  for drawup contingency contract for Cramér–Lundberg model for different starting position of drawup  $z$  (first) and different starting position of drawdown process  $y$  (second). Parameters:  $r = 0.01, \mu = 0.05, \beta = 0.01, \rho = 2.5, a = 10, \alpha = 100$ .

to the drawdown contract without drawup constraints, the fair premium  $p^*$  does not tend to  $\infty$  as  $y \uparrow a$  for the Cramér–Lundberg process (67). This observation is a consequence of the fact that for the Cramér–Lundberg process we have  $W^{(r)}(0) > 0$  and hence  $\lim_{y \rightarrow a^-} \nu(y, z) + \lambda(y, z) \neq 1$  and the denominator in the formula for  $p^*$  given in (43) does not converge to 0.

**5.5. Cancellable drawup contingency for  $a > b$ .** We continue our previous numerical analysis by adding cancellable feature and by considering the insurance contract (44). By Theorem 7 it suffices to find function  $h(y, z, p, \theta^*)$  for the function  $h$  given in (54) and (57) and for the optimal level  $\theta^*$  defined in (58). To calculate (57) for  $a < b$  we use numerical integration. The results are depicted on Figure 7.

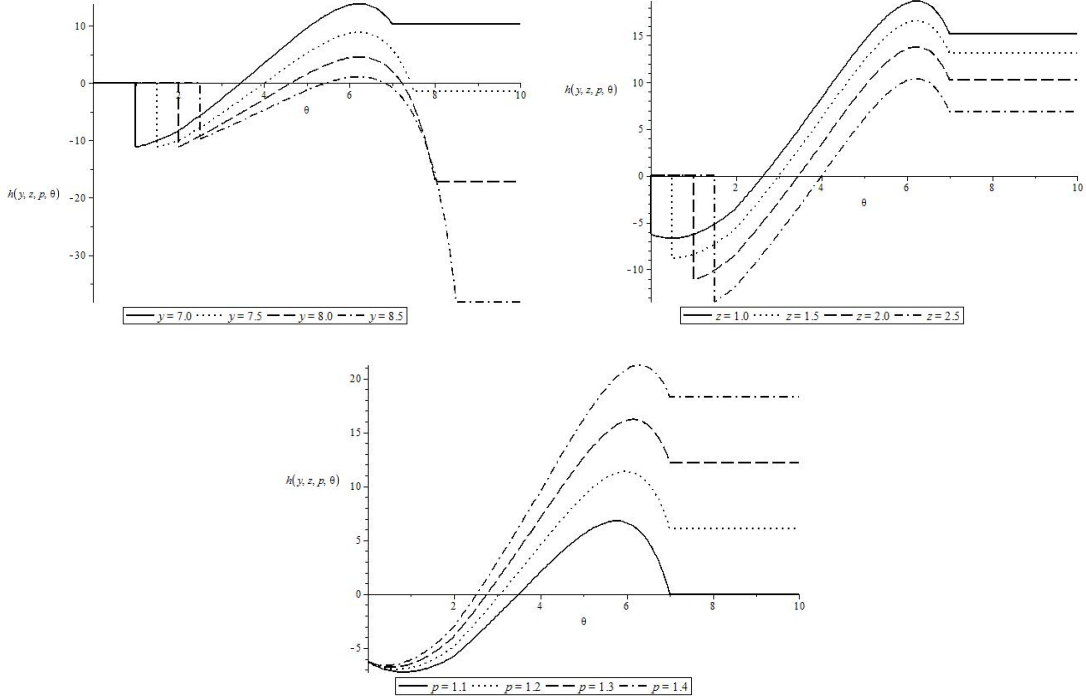


FIGURE 7. The value  $h(y, z, p, \theta)$  for drawup contingency contract for Brownian motion with drift with respect to  $\theta$  for different  $y$  (first),  $z$  (second) and  $p$  (third). Parameters:  $r = 0.01, \mu = 0.03, \sigma = 0.4, a = 10, b = 8, \alpha = 100, c = 50, p = 1.35, y = 7, z = 2$ .

Note that by (58) the optimal drawdown stopping level  $\theta^*$  maximizes function  $h$ . From analyzing the Figure 7, it seems that also for this contract, there is no dependence of optimal  $\theta^*$  on initial positions  $z$  and  $y$  of drawup and drawdown. However, it is clear, that the optimal level of termination is different for the contracts with and without drawup contingency. Even if we take the same parameters, the the existence of new parameter - the staring position of drawup, significantly change the  $\theta^*$  level.

**5.6. Cancellable drawup contingency for  $a = b$ .** We also analyze results for the special case when  $a = b$ . To obtain value of function  $h$  we can use (65). Figure 8 depicts the results for the Cramér-Lundberg model. We can observe the same lack of dependence of the optimal stopping level  $\theta^*$  on the starting position of the drawdown and drawup.

## 6. CONCLUSIONS

In this paper we analyzed few insurance contracts against drawdown and drawup events of log-return of asset price. We model the underlying asset price by a geometric spectrally negative Lévy process. We identified the fair premium  $p^*$  and the optimal stopping rules for the contracts having the cancellable feature. We used the theory of optimal stopping and fluctuation theory of Lévy processes to price these type of contracts.

It is natural to consider other processes for the asset prices than geometric Lévy processes  $S_t = e^{X_t}$ , for example geometric jump-diffusion processes. It is also interesting to do more detailed numerical analysis when jumps are of mixed-exponential type or, more general, they are of phase-type. The idea is to consider all possible shapes of the density of possible jumps in the asset prices. Moreover, one can consider reward  $\alpha$  and fee  $c$  dependent on the process  $X$  observed at the end of the insurance contract. In fact, there is still huge demand for more general insurance contracts that will serve as a policy against large drawdown events. This will be subject of future research.

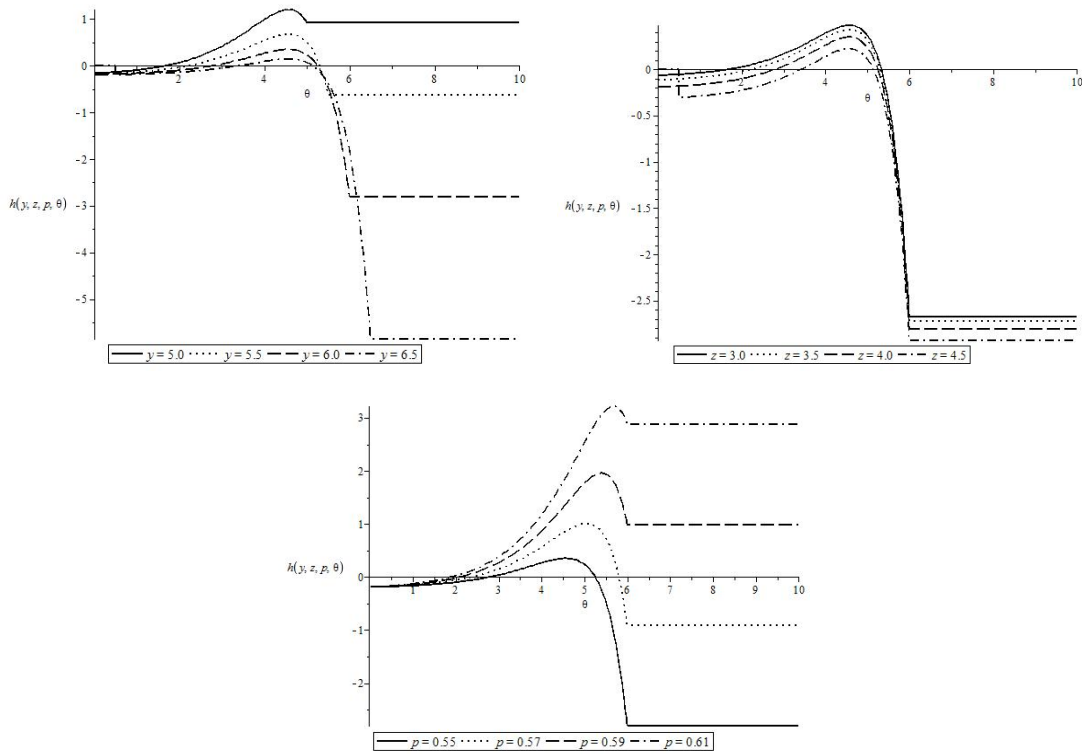


FIGURE 8. The value  $h(y, z, p, \theta)$  for drawup contingency contract for Cramér–Lundber with respect to  $\theta$  for different  $y$  (first),  $z$  (second) and  $p$  (third). Parameters:  $r = 0.01, \mu = 0.04, \beta = 0.1, \rho = 2.5, a = 10, \alpha = 100, c = 50, p = 0.55, y = 6, z = 4$ .

## REFERENCES

- [1] Carr, P., Zhang, H. and Hadjiladis, O. (2011) Maximum drawdown insurance. *International Journal of Theoretical and Applied Finance*, 14(8):1–36.
- [2] Grossman, S. J. and Zhou, Z. (1993) Optimal investment strategies for controlling drawdowns. *Mathematical Finance*, 3(3):241–276.
- [3] Kyprianou, A.E. (2006) *Introductory Lectures on Fluctuations of Lévy Processes with Applications*. Springer.
- [4] Kyprianou, A.E., Kuznetsov, A. and Rivero, V. (2013) The theory of scale functions for spectrally negative Lévy processes. *Lévy Matters II*, Springer Lecture Notes in Mathematics.
- [5] Magdon-Ismaïl, M. and Atiya, A. (2004) Maximum drawdown. *Risk*, 17(10):99–102.
- [6] Mijatović, A. and Pistorius, M.R. (2012) On the drawdown of completely asymmetric Lévy process. *Stochastic Processes and their Applications*, 122(11):3812–3836.
- [7] Nguyen-Ngoc, L. and Yor, M. (2005) Some Martingales Associated to Reflected Lévy Processes. *Séminaire de Probabilités*, XXXVIII:42–69.
- [8] Øksendal, B. and Sulem, A. (2004) *Applied Stochastic Control of Jump Diffusions*. Springer.
- [9] Peskir, G. and Shiryaev, A. (2006) *Optimal Stopping and Free-Boundary Problems*. Birkhäuser.
- [10] Pistorius, M.R. (2004) On Exit and Ergodicity of the Spectrally One-Sided Lévy Process Reflected at Its Infimum. *Journal of Theoretical Probability*, 17(1):183–220.
- [11] Pospisil, L. and Vecer, J. (2010) Portfolio sensitivities to the changes in the maximum and the maximum drawdown. *Quantitative Finance*, 10(6):617–627.
- [12] Sornette, D. (2003) *Why Stock Markets Crash: Critical Events in Complex Financial Systems*. Princeton University Press.
- [13] Vecer, J. (2006) Maximum drawdown and directional trading. *Risk*, 19(12):88–92.
- [14] Vecer, J. (2007) Preventing portfolio losses by hedging maximum drawdown. *Wilmott*, 5(4):1–8.
- [15] Zhang, H. and Hadjiladis, O. (2010) Drawdowns and rallies in finite time-horizon. *Methodology and Computing in Applied Probability*, 12(2):293–308.
- [16] Zhang, H., Leung, T. and Hadjiladis, O. (2013) Stochastic modeling and fair valuation of drawdown insurance. *Insurance: Mathematics and Economics*, 53:840–850.

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